Filtrations and Gradings

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1 Filtered rings and modules

Definition 1.1 (Filtration)**.** A descending sequence $(A_n)_{n\in\mathbb{Z}}$ of ideals of a ring A is called a filtration on A. Similarly, a descending sequence $(M_n)_{n\in\mathbb{Z}}$ of submodules of a module M is called a filtration on M.

Definition 1.2 (Filtered ring, module)**.** A filtered ring is a ring A given with a filtration $(A_n)_{n\in\mathbb{Z}}$ such that

i. $A_0 = A$

ii. $A_p A_q \subset A_{p+q}$

Similarly, a filtered module is a module M given with a filtration $(M_n)_{n\in\mathbb{Z}}$ such that

i. $M_0 = M$

ii. $A_p M_q \subset M_{p+q}$

Example 1.3 (Examples of filtrations)**.**

1. (Trivial filtration) Let A be any ring. Then $A_n = 0$ for $n \geqslant 1$ gives a filtration.
2. ($\mathfrak{m}$-adic filtration) Let $\mathfrak{m}$ be an ideal of a ring A. Then $A_n = \mathfrak{m}^n$ for $n \geqslant 1$ gives a filtration.

Definition 1.4 (Preadditive category)**.** A preadditive category or an Ab-category is a category such that all hom-sets are additive abelian groups and composition is bilinear. That is, for $f, f' \colon a \to b$ and $g, g' \colon a \to b$ we have

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'f$$

Definition 1.5 (Biproduct diagram)**.** Let $\mathcal{C}$ be a category and $a, b \in A$. A biproduct diagram for the objects $a, b \in \mathcal{C}$ is a diagram

$$a \underset{i_1}{\overset{p_1}{\leftrightarrows}} c \underset{i_2}{\overset{p_2}{\rightleftarrows}} b$$

with arrows p_1, p_2, i_1, i_2 such that

$$\begin{aligned} p_1 i_1 &= 1_a \\ p_2 i_2 &= 1_b \\ i_1 p_1 + i_2 p_2 &= 1_c \end{aligned}$$

Definition 1.6 (Additive category)**.** An additive category is an Ab-category with a zero object and a biproduct for each pair of objects.

Definition 1.7 (F_A, additive category of filtered modules)**.** The filtered A modules form an additive category F_A in which morphisms are the filtration compatible homomorphisms (i.e. homomorphisms $u \colon M \to N$ such that $u(M_n) \subset N_n$)

Definition 1.8 (Induced filtration, quotient filtration)**.** If P is a submodule of a filtered A-module M with filtration $(M_n)_{n \in \mathbb{Z}}$ then induced filtration on P is given by $(P_n) = (P \cap M_n)$. Since P is a submodule and M is a filtered ring, this definition indeed satisfies the filtration axioms.

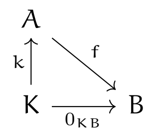
Similarly for a quotient module $N = M/P$ the induced quotient filtration is given by (N_n) where $N_n = (M_n + P)/P$. It is straightforward to check that this filtration is well-defined.

Definition 1.9 (Category theoretic definitions)**.** For a category $\mathcal{C}$ with terminal object 1, a morphism A is injective if for any two global elements $x, y \colon 1 \to A$ of A we have $x = y$ if $f \circ x = f \circ y$. Similarly f is called a surjection if given any global element $y \colon 1 \to B$ of B there exists a global element $x \colon 1 \to A$ such that $y = f \circ x$.

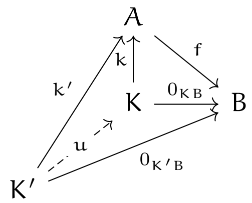
A morphism $f \colon x \to y$ is an epimorphism if for every object z and pair of morphisms $g_1, g_2 \colon y \to z$ we have $g_1 \circ f = g_2 \circ f \implies g_1 = g_2$. (Right cancellation)

A morphism $g \colon x \to y$ is an epimorphism if for every object z and pair of morphisms $f_1, f_2 \colon z \to x$ we have $g \circ f_1 = g \circ f_2 \implies f_1 = f_2$. (Left cancellation)

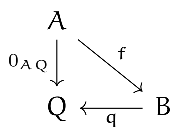
The kernel of a morphism $f \colon A \to B$ in $\mathcal{C}$ is an object K together with a morphism $k \colon A \to A$ such that the following diagram commutes



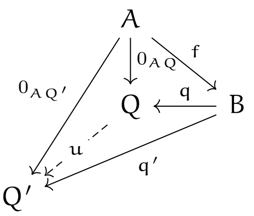
and given any morphism $k' \colon K \to K$ which makes the diagram commute there exists a unique $u \colon K' \to K$ such that the following diagram commutes



Similarly the cokernel of f is an object Q together with a morphism $q \colon B \to Q$ such that the following diagram commutes



Moreover, if there exists another morphism $q' \colon B \to Q'$ making the diagram commute then there exists a unique $u \colon Q \to Q'$ such that the following diagram commutes



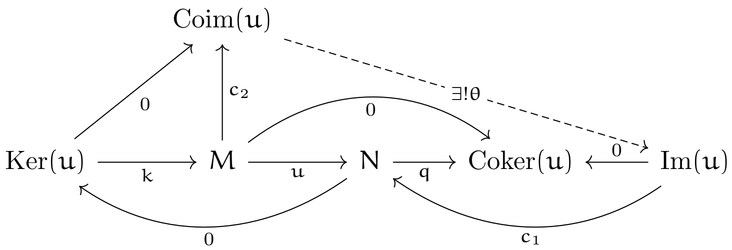
Remark 1.10 (Injectivity and surjectivity in F_A)**.** In F_A these notions of injective and surjective functions correspond to the usual notions. Every morphism $u \colon M \to N$ has a kernel $\mathrm{Ker}(u)$ and a cokernel $\mathrm{Coker}(u)$ corresponding to the usual notions of kernel and cokernel (i.e. $N/\mathrm{Im}(u)$) in modules. We define $\mathrm{Im}(u) = \mathrm{Ker}(N \to \mathrm{Coker}(u))$ and $\mathrm{Coim}(u) = \mathrm{Coker}(\mathrm{Ker}(u) \to M)$

Proposition 1.11. Let $u \colon M \to N$ be a morphism of filtered modules in F_A. Then we have the following factorization

$$\mathrm{Ker}(u) \to M \to \mathrm{Coim}(u) \xrightarrow{\theta} \mathrm{Im}(u) \to N \to \mathrm{Coker}(u)$$

where θ is bijective.

Proof.



Note that $M \to \mathrm{Im}(u)$ vanishes on $\mathrm{Ker}(u)$ thus it factors through the cokernel and we have an isomorphism of modules $\theta \colon \mathrm{Coim}(u) \to \mathrm{Im}(u)$. ☐

Definition 1.12 (Strict morphism)**.** The morphism u in the above proposition is called a strict morphism if θ is an isomorphism of filtered modules.

Remark 1.13. In F_A not all bijections are isomorphisms. Let M be any module with a filtration (M_n) such that $M_{k+1} \subsetneq M_k$ for some $k \in \mathbb{Z}$ and let N be a module such that $N = M$ and filtration (N_n) is given by $N_n = M_n$ except for $N_k = M_{k+1}$. Consider the identity map $i \colon M \to N$. This map is clearly a homeomorphism and an isomorphism of modules yet $i(M_{k+1}) = N_k$.

Proposition 1.14. u in the above proposition is a strict morphism if and only if for all $n \in \mathbb{Z}$ we have $u(M_n) = N_n \cap u(M)$.

Proof. Suppose u maps each M_n to $N_n \cap u(M)$. In any case we have that the coimage is isomorphic as a module to the image. We will show that the induced map is an isomorphism of filtered modules. Let $\overline{u} \colon M/\mathrm{Ker}(u) \to \mathrm{Im}(u)$ be the induced map. This is a straightforward verification since u itself is a morphism. Conversely, suppose that u is a strict morphism. Hence $M/\operatorname{Ker}(u) \cong \operatorname{Im}(u)$ as filtered modules. Thus the induced map $\overline{u}$ is such that $\overline{u}((M_n + \operatorname{Ker} u)/\operatorname{Ker} u) = N_n \cap \operatorname{Im}(u)$. We wish to show that $u(M_n) = N_n \cap \operatorname{Im}(u)$. Let $a \in M_n$. Then $a + \operatorname{Ker}(u) \in (M_n + \operatorname{Ker}(u))/\operatorname{Ker}(u)$ so $a + \operatorname{Ker}(u)$ maps to $u(a) \in N_n \cap \operatorname{Im}(u)$ as desired. Now let $u(a) \in N_n \cap \operatorname{Im}(u)$ which maps to $a + \operatorname{Ker}(u)$ under $\overline{u}$. Say $a = k + t$ for $k \in M_n$ and $t \in \operatorname{Ker}(u)$. Then clearly $u(k) = u(k+t) = u(a)$ since u is a module homomorphism. □

Definition 1.15 (Abelian category)**.** An abelian category A is an Ab-category such that

1. A has a zero object.
2. A has binary biproducts.
3. Every arrow in A has a kernel and a cokernel.
4. Every monic arrow is a kernel, and every epi is a cokernel.

Definition 1.16 (Equalizer)**.** Given morphisms $f, g \colon x \to y$. Their equalizer is

- An object $\operatorname{eq}(f, g) \in \mathcal{C}$
- A morphism $\operatorname{eq}(f, g) \to x$

such that

- Both morphisms f and g become equal when pulled back to eq.
- $\operatorname{eq}(f, g)$ is the universal object with this property.

Proposition 1.17. In an abelian category $\mathcal{C}$ all bijective morphisms are isomorphisms.

Proof. We first note that if the equalizer of any pair of morphisms is an epi then it must be an iso. Since the maps equalized must be equal by the definition of epimorphism and the equalizer must be isomorphic to the identity map between them, and thus an iso. Now note that in an abelian category every monomorphism is the kernel of some map by definition which simply means that it is the equalizer of that map and the zero map. So a monomorphism which is epic would be an isomorphism. □

Remark 1.18. In the monoid of natural numbers, every number considered as a morphism is both monic and epi but the only number with an inverse is 0.

Remark 1.19. With the above counterexample we see that i^{-1} is monic and epi but not an iso. Therefore, F_A is not abelian.

2 Topology defined by a filtration

Definition 2.1 (Equivalent definitions of continuity)**.** Let X, Y be topological spaces with bases $\mathcal{A}, \mathcal{B}$ and let A_x denote the neighborhood basis of each $x \in X$, B_y denote the neighborhood basis of each $y \in Y$. A function $f \colon X \to Y$ is continuous at a point $x \in X$ if

1. For all neighborhoods N of $f(a)$ in Y, $f^{-1}(N)$ is a neighborhood of a.
2. For all $V \in B_{f(x)}$, $f^{-1}(V)$ is a neighborhood of x.
3. For all open sets $\mathcal{O}$ containing $f(a)$, there exists an open set $V \subset X$ containing a such that $V \subset f^{-1}(\mathcal{O})$.

f is a continuous mapping if

1. For all open subsets $\mathcal{O} \subset Y$, $f^{-1}(\mathcal{O})$ is open.
2. f is continuous at x for all $x \in X$.
3. For all $B \in \mathcal{B}$, $f^{-1}(B)$ is open.

Definition 2.2 (Compatible with group structure)**.** Let G be a set with both a group structure and a topology. We say that the topology on G is compatible with the group structure if the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ for $x, y \in G$ are continuous. (Alternatively, $(x, y) \mapsto xy^{-1}$ is continuous.)

Let A be a set with both a ring structure and a topology. We say that the topology on A is compatible with the ring structure if the maps $(x, y) \mapsto x - y$ and $(x, y) \mapsto xy$ for $x, y \in A$ are continuous.

Let M be a set with both an A-module structure and a topology. We say that the topology on M is compatible with the module structure if the maps $(r, m) \mapsto rm$ and $(m, n) \mapsto m - n$ for $r \in A$ $m, n \in M$ are continuous.

Definition 2.3 (Topological group, ring, module)**.** Let G be a set with both a group structure and a topology compatible with the group structure. Then G is a topological group.

Let A be a set with both a ring structure and a topology compoatible with the ring structure. Then A is a topological ring.

Let M be a set with both a module structure and a topology compatible with the module structure. Then M is a topological module.

Lemma 2.4. Let G be a topological abelian group. Then the translation map $T_a(x) = x + a$ is a homeomorphism.

Proof. Note that T_a is the composition of $x \mapsto (x, a)$ and $(x, y) \mapsto x + y$ which are both continuous and thus T_a is also continuous. Moreover note that T_{-a} is the inverse of a which is also continuous by a similar argument. This finishes the proof. □

Definition 2.5 (Neighborhood)**.** Let X be a topological space. A subset N of X is called a neighborhood of a point $x \in X$ if there exists an open $U \subset X$ such that $x \in U \subset N$.

Definition 2.6 (Neighborhood system)**.** Let X be a topological space. A collection of neighborhoods of a point is called a neighborhood system of that point.

Definition 2.7 (Neighborhood basis)**.** Let X be a topological space. A neighborhood system $\mathfrak{B}$ for a point $x \in X$ is called a neighborhood basis if for any neighborhood N of x there exists a neighborhood P in $\mathfrak{B}$ such that $P \subset N$.

Lemma 2.8. Let G be a topological abelian group. If N is a neighborhood of 0 then $x + N$ is a neighborhood of $x \in G$ and every neighborhood of x is of this form.

Proof. By the previous lemma we know that given $x \in G$ the map T_x is a homeomorphism. Let N be a neighborhood of 0. Then there exists an open $U \subset G$ such that $0 \in U \subset N$. Then $T_x(U) = x + U$ is open. So $x + N$ is a neighborhood of x since $x \in x + U \subset x + N$. Conversely, let M be a neighborhood of x. So there exists an open $V \subset G$ such that $x \in V \subset M$. Since T_{-x} is a homeomorphism, $T_{-x}(V) = -x + V$ is also open. But note that $0 \in -x + V \subset -x + M$ and thus $M = x + (-x + M)$ where $-x + M$ is a neighborhood of 0, as desired. □

Corollary 2.9. Let G be a topological abelian group. If $\mathfrak{B}$ is a neighborhood basis for 0 then $\{x + V \mid V \in \mathfrak{B}\}$ is a neighborhood basis for $x \in G$.

Proof. Let N be a neighborhood of $x \in G$. Then $N = x + M$ where M is a neighborhood of 0 by the lemma. Since $\mathfrak{B}$ is a neighborhood basis for 0 there exists $V \in \mathfrak{B}$ such that $V \subset M$. But then $x + V \subset x + M$ as desired. □

Proposition 2.10 (Continuity of a homomorphism)**.** Let G and G' be topological abelian groups and $f\colon G \to G'$ be a homomorphism of groups. Then f is continuous if and only if it is continuous at 0.

Proof. If f is continuous it is in particular continuous at 0. Suppose f is continuous at 0. Let $g \in G$ and let $g + N$ be a neighborhood of g where N is a neighborhood of 0. By continuity at 0 there exists a neighborhood V of 0 such that $f(V) \subset N$. Then $f(g + V) \subset f(g) + N$ since for all $v \in V$ we have $f(g + v) = f(g) + f(v) \in f(g) + V \subset f(g) + N$. □

Lemma 2.11. Let X be a topological space. A set $U \subset X$ is open if and only if U is a neighborhood of each of its elements.

Proof. Obvious. □

Corollary 2.12. Let X be a topological space. A set U is open if and only if for every $x \in U$ there exists an element V in the neighborhood basis of x such that $V \subset U$

Proof. Note that the neighborhoods of a point $x \in X$ is given by

$$N(x) = \{P \subset X \mid \exists V \text{in the neighborhood basis such that } V \subset P\}$$

The proof follows easily from the previous lemma and this definition. □

Proposition 2.13 (Topology given by a filtration)**.** Let M be a filtered A-module with filtration $(M_n)_{n\in\mathbb{Z}}$. Then this filtration induces a unique topology on M compatible with the group structure such that the M_n are a neighborhood basis for 0.

Proof. (Uniqueness) First note that by Corollary 2.8. we know the neighborhood basis of each $x \in M$ since we know the neighborhood basis of 0. Then by Corollary 2.10. open sets are completely determined by the neighborhood bases. Hence the uniqueness follows.

(Existence) We define the topology τ on M to be such that

$$U \text{ is open } \iff \forall x \in U\ \exists n \in \mathbb{Z} \text{ such that } x + M_n \subset U$$

Now we check the topology axioms:

Clearly $\varnothing$ is vacuously an open set and M is also an open set since $x+M_n \subset M$ for all $n \in \mathbb{Z}$ and $x \in M$.

Let $S = \bigcap_{i=0}^{k} U_i$ be a finite intersection of open sets. Let $x \in S$. Then $x \in U_i$ for all i. So there exists $n_i \in \mathbb{Z}$ for each U_i such that $x + M_{n_i} \subset U_i$. Then note that $x + \bigcap_{i=0}^{k} M_{n_i} \subset \bigcap_{i=0}^{k} U_i$ since $\bigcap_{i=0}^{k} \subset M_{n_i}$. Picking $z = \max\{n_0, n_1, \dots, n_k\}$ we attain

$$x + M_z \subset x + \bigcap_{i=0}^{k} M_{n_i} \subset S$$

where the first equality follows since $M_z = \bigcap_{i=0}^{k} M_{n_i}$ as M_n's give a descending filtration.

Now let $Q = \bigcup_{i \in I} U_i$ be an arbitrary union of open sets and $x \in Q$. Then $x \in U_i$ for some $i \in I$ and so $x + M_n \subset U_i$. Note that $U_i \subset Q$ and so $x + M_n \subset U_i \subset Q$.

(Neighborhood basis) Now we show that M_n's form a neighborhood basis of 0 with this topology. First note that each M_n is open since for all x in M_n we have $x + M_n = M_n \subset M_n$ since M_n is a subgroup. Then M_n's are neighborhoods of 0 since $0 + M_{n+1} = M_{n+1} \subset M_n$ and M_{n+1} is open. To show that these M_n's form a neighborhood basis: Let N be a neighborhood of 0. Then there exists an open U such that $0 \in U \subset N$. Since U is open and $0 \in U$, there exists $n \in \mathbb{Z}$ such that $0 + M_n = M_n \subset U$ as desired.

(Compatibility) Now we will show that the topology we have defined is compatible with the group structure on M, i.e., the map $(x, y) \mapsto x - y$ is continuous. Let $a, b \in M$. Put $x = a + u$ and $y = b + v$. We wish to show that $x - y$ is as near as we please to $a - b$ whenever u and v are close enough to 0. Let N be a neighborhood of 0. Note that $(x - y) - (a - b) = u - v$ and there exists $n \in \mathbb{Z}$ such that $M_n \subset N$ since M_n's form a neighborhood basis of 0. If we choose $u, v \in M_n$ then $u - v \in M_n$ so we have that $x - y - (a - b) \in M_n$ and thus $x - y \in (a - b) + M_n \subset (a - b) + N$ as desired. □

Corollary 2.14. Taking $M = A$, A becomes a topological ring and in general M becomes a topological A-module.

Proof. (Topological ring) We only need to check the compatibility of multiplication as the compatibility of the group structure is checked in the proposition. Let $a, b \in A$. Put $x = a + u$ and $y = b + v$. We wish to show that we can get xy as near as we please to ab whenever u and v are close enough to 0. Let N be a neighborhood of 0. Note that $xy - ab = ab + av + uv$ and there exists $n \in \mathbb{Z}$ such that $M_n \subset N$ since M_n's are neighborhood basis of 0. Then if we choose $u, v \in M_n$ we attain $ab, av, uv \in M_n$ since M_n is an ideal. Then $xy - ab \in M_n$ so $xy \in ab + M_n \subset ab + N$ as desired.

(Topological module) We only check the continuity of scalar multiplication. Let $r \in A$ and $m \in M$. Put $x = r + u$ and $y = m + v$. We wish to show that we can get xy as close as we want to rm whenever u and v are close enough to 0. Let N be a neighborhood of 0. Note that $xy - rm = um + uv + rv$ and there exists $n \in \mathbb{Z}$ such that $M_n \in N$ since M_ns are a neighborhood basis of 0. Then if we choose $m, v \in M_n$ we attain $um, uv, rv \in M_n$ since M_n is a submodule. Then $xy - rm \in M_n$ so $xy \in rm + M_n \subset rm + N$ as desired. □

Remark 2.15. Let M and N be two filtered A-modules with filtrations (M_n) and (N_n) and $u\colon M \to N$ be a morphism of modules. We can show that if $u(M_n) = N_n$ and u is an isomorphism of modules then u is continuous. Indeed, let O be an open subset of N. We will show that $u^{-1}(O)$ is open. Let $x \in u^{-1}(O)$. Then $u(x) \in O$ and since O is open there exists $k \in \mathbb{Z}$ such that $u(x) + M_k \subset O$. Applying the inverse to both sides we attain $x + N_k \subset u^{-1}(O)$ which shows that the inverse image is open.

Remark 2.16 ($\mathfrak{m}$-adic topology)**.** If $\mathfrak{m}$ is an ideal of A then the $\mathfrak{m}$-adic topology on an A-module M is the topology defined by the $\mathfrak{m}$-adic filtration of M.

Proposition 2.17 (Hausdorff $\iff$ diagonal closed)**.** Let X be a topological space. Then X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is closed.

Proof. Suppose X is Hausdorff. We wish to show that $S = X \times X \setminus D$ is open. Let $(x, y) \in S$. Then $x \neq y$ so there exists open U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. Then $U \cap V \subset X \times X \setminus D$ and $(x, y) \in U \times V$. Conversely, suppose that Δ is closed. Let $x, y \in X$ such that $x \neq y$. Then $(x, y) \in X \times X \setminus \Delta$ which is open. So there exists an basic open set $Z \subset X \times X \setminus \Delta$ such that $(x, y) \in Z$. By the properties of the product topology $Z = U \times V$ for some open $U, V \subset X$. Finally note that $U \cap V = \varnothing$ since otherwise $Z \cap \Delta = (U \times V) \cap \Delta \neq \varnothing$. □

Proposition 2.18 (Closure of submodules)**.** Let N be a submodule of a filtered module M. Then the closure$\overline{N}$ of N is equal to $\bigcap(N + M_n)$.

Proof. Since the neighborhoods of $x \in M$ correspond to $x+M_n$ have by definition of closure

$$\begin{aligned}
x \notin \overline{N} &\iff \exists k \in \mathbb{Z} \text{ s.t. } (x + M_n) \cap N = \varnothing \\
&\iff \nexists z \in N \text{ s.t. } z = x + t \text{ for any } t \in M_k \text{ for some } k \\
&\iff \nexists z \in N \text{ s.t. } z = x - t \text{ for any } t \in M_k \text{ for some } k \\
&\iff x \neq n + t \text{ for any } n \in N, t \in M_k \text{ for some } k \\
&\iff x \notin N + M_k \text{ for some } k \\
&\iff x \notin \cap(N + M_n)
\end{aligned}$$

Alternatively, show that $x \notin N + M_k \iff (x + M_k) \cap N = \varnothing$. Suppose first that $(x + M_k) \cap N = \varnothing$ but $x \in N + M_k$. Then $x = n + r$ for some $n \in N$, $r \in M_k$. Then $x + M_k = n + M_k$ contains $n \in N$, a contradiction.

Conversely suppose that $x \notin N + M_k$ but $(x + M_k) \cap N \neq \varnothing$. Then let $z \in (x+M_k) \cap N$ so $z = x+t$ for some $t \in M_k$ and $z \in N$. So $x = z-t \in N+M_k$, a contradiction. □

Corollary 2.19 (Hausdorff $\iff$ 0 is closed)**.** M is Hausdorff if and only if $\bigcap M_n = 0$

Proof. Note that $\bigcap M_n = \overline{\{0\}}$. If M is Hausdorff then it is T_1 and so singletons are closed. Suppose now that $\{0\}$ is closed. We know that M is Hausdorff iff the diagonal $\Delta = \{(m, m) \mid m \in M\}$ is closed in $M \times M$. But note that $f(x, y) = x - y$ is continuous so $f^{-1}(\{0\}) = \Delta$ is closed. □

Remark 2.20 ($T_0 \iff T_1 \iff T_2$)**.** Note that if M is T_0 then $\{0\}$ is again closed. Let $g \neq 0$ we wish to show that $G \setminus \{0\}$ is a neighborhood of g. Since M is T_0 either 0 or g have a neighborhood not containing the other. If g has a neighborhood not containing 0 then we are done. Suppose 0 has a neighborhood, N not containing g. Then $-g + N$ is a neighborhood of $-g$ not containing 0. Applying the additive inverse map we see that $g + N$ is a neighborhood of g not containing 0. Thus $\{0\}$ is closed. Then we have the result by the corollary and implications $T_2 \implies T_1 \implies T_0$.

Proposition 2.21 (Hausdorff $\iff$ unique limits)**.** Let M be a filtered module. Then convergent sequences in M have unique limits if and only if M is Hausdorff.

Proof. Suppose (x_n) is a sequence in Hausdorff M such that $x_n \to x_1$ and $x_n \to x_2$ where $x_1 \neq X_2$. Then for all $n \in \mathbb{Z}$ there exists $s_1, s_2 \in \mathbb{Z}$ such that

$$\begin{aligned} x_k &\in x_1 + M_n \text{ for } k \geqslant s_1 \\ x_k &\in x_2 + M_n \text{ for } k \geqslant s_2 \end{aligned}$$

Then for $k \geqslant \max\{s_1, s_2\}$ we have $x_k \in (x_1 + M_n) \cap (x_2 + M_n) = \varnothing$, a contradiction.

Conversely, suppose that each convergent sequence has a unique limit but there exists $x \neq y$ in M such that there exists $n, k \in \mathbb{Z}$ such that $(x + M_n) \cap (y + M_k) \neq \varnothing$. Construct a sequence (a_z) as follows:

$$\begin{aligned} a_0 &= k_0 \in (x + M_n) \cap (y + M_k) \\ a_1 &= k_1 \in (x + M_{n+1}) \cap (y + M_{k+1}) \\ &\vdots \\ a_z &= k_z \in (x + M_{n+z}) \cap (y + M_{k+z}) \end{aligned}$$

Then any neighborhood of x and y contain an element of the sequence. Hence both are limit points. □

3 Completion of filtered modules

Definition 3.1 (Cauchy sequence)**.** Let G be a topological abelian group. A sequence (x_ν) is called Cauchy if for every neighborhood U of 0 in G there exists an integer $S(U)$ such that

$$x_\nu - x_\mu \in U \text{ for all } \nu, \mu \geqslant s(U)$$

Two Cauchy sequences (x_ν) and (y_ν) are called equivalent if $x_\nu - y_\nu \to 0$.

Remark 3.2. A topological space is called first countable if every point has a countable neighborhood basis. A Cauchy net is a map from a directed set A to a topological space such that for every entourage V there exists $c \in A$ such that $(x_a, x_b) \in V$ for $a, b \geqslant c$. In such a topological space we have a Cauchy sequence converging to x for every Cauchy net converging to x (and vice versa).

Since our filtered modules will be first countable spaces, we will only consider sequences instead of nets by the above correspondence.

Remark 3.3 (Equivalence)**.** In the topology given by a filtration two Cauchy sequences (x_ν) and (y_ν) are equivalent if and only if for all M_n there exists some $s(M_n) \in \mathbb{Z}$ such that $x_\nu - y_\nu \in M_n$.

Proof. Suppose (x_ν) and (y_ν) are equivalent. Then for all neighborhoods of 0 we have $x_\nu - y_\nu \to 0$, in particular for M_n's. Now suppose for all M_n there exists some $s(M_n) \in \mathbb{Z}$ such that $x_\nu - y_\nu \in M_n$. Let U be a neighborhood of 0. Since M_n's form a fundamental set of neighborhoods there exists $k \in \mathbb{Z}$ such that $M_k \subset U$. Then there exists $s(M_k)$ such that $x_\nu - y_\nu \in M_k \subset U$ for $\nu \geqslant s(M_k)$, as desired. □

Definition 3.4 (Hausdorff completion for countable neighborhood basis)**.** Let M be a filtered A-module. When we have a countable neighborhood basis for 0 we can use Cauchy sequences to construct the completion instead of nets and this is always the case with our filtration setup. The completion $\hat{M}$ of module M is defined to be the set of equivalence classes of Cauchy sequences in M. Note that the equivalence class of $(x_\nu - y_\nu)$ only depend on the equivalence classes of (x_ν) and (y_ν). Moreover, scalar multiplication respect the Cauchy property of sequences. Then we have a well defined module structure on $\hat{M}$.

Proposition 3.5 (Filtration of a Hausdorff completion)**.** Let M be a filtered A-module with completion $\hat{M}$. Then $\hat{M}$ becomes a filtered $\hat{A}$-module if we define

$$\hat{M}_n = \mathrm{Ker}(\hat{M} \to M/M_n)$$

Proof. (i.) We have $\hat{M}_0 = \hat{M}$

(ii.) $M_{n+1} \subset M_n \implies \hat{M}_{n+1} \subset \hat{M}_n$

(iii.) Let $[(a_n)] \in A_q$ and $[(m_n)] \in M_p$. $[(a_n)(m_n)]$ maps to 0 under $\hat{M} \to M/M_n$. Note that the value of $[(a_n)(m_n)]$ under this map is just the eventual constant value of $(a_n)(m_n)$ in M/M_{p+q} but $A_pM_q \subset M_{p+q}$ and so $(a_n)(m_n) = (a_n m_n) \subset M_{p+q}$ and then this product of sequences is eventually 0 in M/M_{p+q} showing that $A_pM_q \subset M_{p+q}$.

Thus this definition makes $\hat{M}$ a filtered $\hat{A}$-module. □

Remark 3.6. We note that $\hat{M}_n$ is then the completion of M_n with the above definition. Indeed, take an equivalence class $[(x_\nu)]$ from the completion of M_n. Then by definition (x_ν) is in M_n and its constant value in M/M_n is $0 + M_n$ and thus it is in the kernel. Conversely, let $[(x_\nu)]$ be an equivalence class in the kernel. Then the eventual value of (x_ν) in M/M_n is $0 + M_n$. Thus (x_ν) falls in M_n eventually. Suppose the first z many terms of (x_n) are not in M_n. Define a sequence (y_μ) by setting $y_k = 0$ if $k < z$ and $y_k = x_k$ if $k \geqslant z$. Then (y_μ) is Cauchy sequence in M_n and it is equivalent to (x_ν).

Proposition 3.7 (Map to completion)**.** Let M be a filtered A-module. Let $\phi\colon M \to \hat{M}$. Then $\operatorname{Ker}\phi = \bigcap M_n$.

Proof.

$$\begin{aligned} p \in \bigcap M_n &\iff p - 0 = (p_\nu) - (0_\nu) \in M_n \text{ for all } n \in \mathbb{Z} \\ &\iff (p_\nu) \text{ and } (0_\nu) \text{ are equivalent} \\ &\iff \phi(p) = [(p_\nu)] = [(0_\nu)] \\ &\iff p \in \operatorname{Ker}\phi \end{aligned}$$

□

Remark 3.8 (Topology of completion)**.** We note that the topology of $\hat{M}$ given above is more explicitly expressed to be composed of open sets $\hat{U}$ corresponding to the open sets U in M such that

$$\hat{U} = \{[(x_n)] \in \hat{M} \mid \forall (y_n) \sim (x_n), \quad y_n \in U \text{ ultimately}\}$$

Definition 3.9 (Inverse limit)**.** A sequence of groups $\{A_n\}$ together with homomorphisms

$$\theta\colon A_{n+1} \to A_n$$

is called an inverse system. The group of all coherent sequences (a_n) (i.e. $a_n \in A_n$ and $\theta_{n+1}a_{n+1} = a_n$) is called the inverse limit of the system.

Proposition 3.10 (Isomorphism of Hausdorff completion to inverse limit)**.** Let M be a filtered A-module and $\hat{M}$ be its completion. Then $\hat{M} \cong \varprojlim M/M_n$.

Proof. Suppose (x_ν) is a Cauchy sequence in M. Then the image of x_ν in M/M_n is ultimately constant since there exists $s(M_n) \in \mathbb{Z}$ such that $x_\nu - x_\mu \in M_n$ for $\nu, \mu \geqslant s(M_n)$ and hence the image of the difference is $0 + M_n$ in M/M_n. Call this constant ξ_n. Consider the projection

$$G/G_{n+1} \xrightarrow{\theta_{n+1}} G/G_n$$

Then $\xi_{n+1} \mapsto \xi_n$ under θ_{n+1}. Thus a Cauchy sequence (x_ν) in M defines a coherent sequence (ξ_n).

Also note that equivalent Cauchy sequences define the same sequence (ξ_n) since their difference is eventually zero in the quotient module. Moreover, we can construct a Cauchy sequence (x_n) giving rise to (ξ_n) simply by letting $x_n \in \xi_n$ (thus giving $x_{n+1} - x_n \in M_n$).

Finally, note that sequences giving rise to a (ξ_n) are equivalent. Let (x_ν) and (y_ν) be two sequences which are associated to the same (ξ_n). Then $x_\nu - y_\nu$ is ultimately zero in M/M_n since these sequences are ultimately equal to the same constant. Thus $x_\nu - y_\nu \in M_n$ ultimately, showing that the sequences are equivalent.

The coherent sequences (ξ_n) define an A-module in an obvious way.

This setup gives us a bijection ϕ, mapping equivalence classes of Cauchy sequences in $\hat{M}$ to coherent sequences in $\varprojlim M/M_n$. It is a routine check to show that this map is a module isomorphism.

Additionally we check that the map is a morphism of filtered modules.

Define a filtration $(N_n)_{n\in\mathbb{Z}}$ on $\varprojlim M/M_n$ where $N_n = \mathrm{Ker}(\varprojlim M/M_n \to M/M_n)$. We will show that $\phi(\hat{M}_n) = N_n$.

$$\begin{aligned}
[(x_\nu)] \in \hat{M}_n &\iff [(x_\nu)] \in \mathrm{Ker}(\hat{M} \to M/M_n) \\
&\iff \exists n \in \mathbb{Z}\ x_\nu + M_n = 0 + M_n \text{ for all } \nu \geqslant n \\
&\iff \phi([[(x_\nu)]]) = 0 \\
&\iff \phi([[(x_\nu)]]) \in \mathrm{Ker}(\varprojlim M/M_n \to M/M_n) \\
&\iff \phi([[(x_\nu)]]) \in N_n
\end{aligned}$$

Then since ϕ is a bijection $\phi^{-1}(N_n) = M_n$. So both ϕ and ϕ^{-1} are morphisms of filtered modules and hence ϕ is an isomorphism. □

Definition 3.11 (Exact sequence of inverse systems)**.** Suppose that $\{A_n\}, \{B_n\}, \{C_n\}$ are three inverse systems and we have commutative diagrams of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0
\end{array}$$

Then we say that we have an exact sequence of inverse systems.

Remark 3.12. The diagram above induces homomorphisms (it is easily checked that the commutative diagrams induce morphisms of coherent sequences)

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

but this sequence is not always exact.

Proposition 3.13 (Snake lemma)**.** Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{\nu} & M'' & \longrightarrow & 0 \\
 & & {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle f''}\downarrow & & \\
0 & \longrightarrow & N' & \xrightarrow[u']{} & N & \xrightarrow[\nu']{} & N'' & \longrightarrow & 0
\end{array}$$

be a commutative diagram of A-modules and homomorphisms with the arrows exact. Then there exists an exact sequence

$$\begin{aligned}
0 \to \mathrm{Ker}(f') \to \mathrm{Ker}(f) \to &\mathrm{Ker}(f'') \\
&\xrightarrow{d} \mathrm{Coker}(f') \to \mathrm{Coker}(f) \to \mathrm{Coker}(f'') \to 0
\end{aligned}$$

Proof. If $x'' \in \mathrm{Ker}(f'')$ we have $x'' = \nu(x)$ for some $x \in M$ and $\nu'(f(x)) = f''(\nu(x)) = 0$. Then $f(x) \in \mathrm{Ker}(\nu') = \mathrm{Im}(u')$ so $f(x) = u'(y')$ for some $y' \in N'$. Thus define $d(x'')$ to be the image of y' in $\mathrm{Coker}(f')$. □

Proposition 3.14 (Exactness of inverse limit)**.** If $0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ is an exact sequence of inverse systems then

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n$$

is always exact. If $\{A_n\}$ is a surjective system then

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to 0$$

is exact.

Proof. Let $A = \prod_{n=1}^{\infty} A_n$ and define $d^A : A \to A$ by $d^A(a_n) = a_n - \theta_{n+1}(a_{n+1})$. Note that $\operatorname{Ker} d^A = \varprojlim A_n$. Similarly, define d^B and d^C. Then the exact sequence of inverse systems give a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow d^A & & \downarrow d^B & & \downarrow d^C & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

By snake lemma we see that

$$\begin{aligned} 0 \to \operatorname{Ker}(d^A) \to \operatorname{Ker}(d^B) \to & \operatorname{Ker}(d^C) \\ & \to \operatorname{Coker}(d^A) \to \operatorname{Coker}(d^B) \to \operatorname{Coker}(d^C) \to 0 \end{aligned}$$

is exact. Now we note that the surjectivity of $\{A_n\}$ implies the surjectivity of d^A since then there is a coherent sequence to each element so we only need to inductively solve

$$x_n - \theta_{n+1}(x_{n+1}) = a_n$$

for $x_n \in A_n$ given $a_n \in A_n$. □

Proposition 3.15 (Exactness of completion)**.** Let $0 \to M' \to M \xrightarrow{p} M'' \to 0$ be an exact sequence of modules. Let M have the topology defined by a sequence $\{M_n\}$ of subgroups and give M' and M'' induced topologies via $\{M'_n \cap M_n\}$ and $\{pM_n\}$. Then

$$0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$$

is exact.

Proof. We apply the previous proposition to the exact sequences

$$0 \to \frac{M'}{M' \cap M_n} \to \frac{M}{M_n} \to \frac{M''}{pM_n} \to 0$$

□

Corollary 3.16 (Completion quotient isomorphic to quotient)**.** Let M a filtered A-module and $\hat{M}$ be its completion. Then $\hat{M}_n$ is a submodule of $\hat{M}$ and we have

$$\hat{M}/\hat{M}_n \cong M/M_n$$

Proof. Apply the previous proposition with $M' = M_n$. Note that $M'' = M/M_n$ has the discrete topology (since $\{pM_n\} = \{0\}$ all subsets are open) so $\hat{M}'' = M''$. The result follows.

Alternatively, since M/M_n is the coimage and $\hat{M}/\hat{M}_n$ is the image of $M \xrightarrow{\phi} \hat{M}$, it is enough to check that ϕ is a strict morphism, that is, $\phi(M_n) = \hat{M}_n \cap \phi(M)$ for all $n \in \mathbb{Z}$.

Let $x \in M_n$. Then clearly the equivalence class of constantly x is both in the image and the completion of M_n. Conversely, let $[(x_\nu)] \in \hat{M}_n \cap \phi(M)$. So there exists some $x \in M$ such that $\phi(x) = [(x_\nu)]$. Thus $(x) \in [(x_\nu)]$ and $x - x_\nu \to 0$. Then there exists $s(M_n) \in \mathbb{Z}$ such that $x - x_\nu \in M_n$ for $\nu \geqslant s(M_n)$ so $x \in x_\nu + M_n = M_n$ since $x_\nu \in M_n$. □

Definition 3.17 (Complete)**.** Let M be a filtered A-module with completion $\hat{M}$. M is called complete if $\phi\colon M \to \hat{M}$ is a surjection.

Corollary 3.18. Let M be a filtered A-module with completion $\hat{M}$. Then $\hat{\hat{M}} \cong \hat{M}$

Proof. Take inverse limits of both sides in the previous corollary. □

Proposition 3.19 (Completion is Hausdorff)**.** Let N be a filtered A-module and M be its Hausdorff completion. Then M is Hausdorff.

Proof. Note that $\phi^{-1}\colon \hat{M} \to M$ is an isomorphism. Then by the first isomorphism theorem

$$\hat{M} \Big/ \bigcap \hat{M}_n \cong M \cong \hat{M}$$

□

Proposition 3.20. If a sequence (x_n) converges to some $c \in M$ then $(x_{n+1} - x_n)$ converges to zero.

Proof. Obvious. □

Proposition 3.21 (Cauchy Criterion)**.** A filtered module M is complete if and only if every Cauchy sequence in M converges.

Proof. Let $\phi\colon M \to \hat{M}$ denote the canonical homomorphism. Suppose that all Cauchy sequences in M converge. Let $[(x_\nu)] \in \hat{M}$. Then $x_\nu \to x$ for some $x \in M$ and so for all $n \in \mathbb{Z}$ we have $x_\nu \in x + M_n$ eventually and thus $x_\nu - x \in M_n$ showing that $x_\nu - x \to 0$ and $\phi(x) = [(x)] = [(x_\nu)]$. Conversely suppose that ϕ is surjective. Let $[(x_\nu)]$ be a Cauchy sequence in M. Since ϕ is surjective there exists x in M such that $\phi(x) = [(x)] = [(x_\nu)]$. Then for all $n \in \mathbb{Z}$ we have $x_\nu - x \in M_n$ eventually and thus $x_\nu \in x + M_n$ eventually, showing that $x_\nu \to x$. □

Proposition 3.22 (Quotient of complete is complete)**.** Let M be a complete filtered module. Then if N is a submodule of M, M/N is complete.

Proof. Let $(x_n + N)$ be a Cauchy sequence in M/N. We will construct a Cauchy sequence (y_n) in M such that $(x_n + N) = (y_n + N)$. Set $x_1 = y_1$. Suppose y_n has been chosen such that $y_n + N = x_n + N$. Note that since $(x_n + N)$ is Cauchy for all $k \in \mathbb{Z}$ we have $x_{n+1} - x_n + N \in (M_k + N)/N$ eventually. So $x_{n+1} - x_n + N = p + j + N$ for some $p \in M_k$ and $j \in N$. Thus $x_{n+1} - x_n - p - j \in N$. Then for some $u \in N$ we have $x_{n+1} - x_n + u \in M_k$ eventually. Choose $y_{n+1} = x_{n+1} - (x_n - y_n) + u$. Then $y_{n+1} + N = x_{n+1} + N$ and $y_{n+1} - y_n = x_{n+1} - x_n + u \in M_k$ eventually so (y_n) is Cauchy and since M is complete (y_n) converges to some $y \in M$. Then $\lim(x_n + N) = \lim(y_n + N) = y + N$. □

Proposition 3.23 (Series convergence in complete Hausdorff filtered module)**.** Let M be a filtered module, Hausdorff and complete. A series $\sum x_n$ where $x_n \in M$ converges in M if and only if its general term x_n converges to zero.

Proof. Left to right direction follows directly from the above proposition. Now suppose x_n tends to zero in M. Then for all $p \in \mathbb{Z}$ there exists $n(p) \in \mathbb{Z}$ such that for $n \geqslant n(p)$ we have $x_n \in M_p$. Then for $n \geqslant n(p)$ and for all $k \geqslant 0$ we have $x_n + x_{n+1} + \cdots + x_{n+k} \in M_p$. Thus the sequence of partials sums of $\sum x_n$ is Cauchy and the result follows from the Cauchy criterion. $\square$

Remark 3.24 (Product topology)**.** Let $\{X_i\}_{i \in I}$ be a family of topological spaces. We note that in the product topology $\prod_{i=1} X_i$ the sets of the form

$$\prod_{i \in I} U_i$$

such that U_i's are open and $U_i = X_i$ except for finitely many $i \in I$ form a basis.

Also, supposing that Y is a topological space, and for each $i \in I$ we have a function $f_i : Y \to X_i$. Then the function $f : Y \to \prod_{i \in I} X_i$ defined by $f(y) = \langle f_i(y) \rangle_{i \in I}$ is continuous iff each f_i is.

Finally note that the product topology is Hausdorff (resp. complete) iff each product is Hausdorff (resp. complete).

Proposition 3.25 (The ring of formal power series is Hausdorff and complete)**.** Let A be a ring and $\mathfrak{m}$ be an ideal of A. If A is Hausdorff and complete for the $\mathfrak{m}$-adic topology, the ring of formal power series $A[[X]]$ is Hausdorff and complete for the $(\mathfrak{m}, X)$-adic topology.

Proof. We first show that the elements in $(\mathfrak{m}, X)^n$ have the form

$$a_0 + a_1 X + a_2 X^2 + \cdots + a_k X^k + \cdots$$

where $a_p \in \mathfrak{m}^{n-p}$ for $0 \leqslant p \leqslant n$ by induction on n.

The base case is clear. Suppose the elements in $(\mathfrak{m}, X)^n$ have the above form. Let

$$p(x) = a_0 + a_1 X + \cdots \in (\mathfrak{m}, X)^n$$
$$q(x) = b_0 + b_1 X + \cdots \in (\mathfrak{m}, X)$$

Then for $k \in \{0, \ldots, n+1\}$ the kth coefficient of $p(X)q(X)$ is $\sum_{i=0}^{k} a_i b_{k-i}$ but then for each $a_i b_{k-1}$ we have that $a_i b_{k-1} \in \mathfrak{m}^{n+1-i}$ thus $\sum_{i=0}^{k} a_i b_{k-i} \in \mathfrak{m}^{n+1-k}$ as desired. Now note that the map $\phi \colon A[[X]] \to A^{\mathbb{N}}$ mapping $\sum_{n=0}^{\infty} a_i X^i$ to $(a_i)_{i \in \mathbb{N}}$ is a group isomorphism. Note that with this filtration $A[[X]]$ is a topological group and A^N is a topological group since A is a topological group with the $\mathfrak{m}$-adic topology. Finally we will show that this map is a homeomeorphism. It is enough to check continuity at zero. To this end, let O be an a basic open set containing 0 in $A^{\mathbb{N}}$ then $O = \prod_{i=1}^{\infty} U_i$ for open U_i where for all but finitely many $i \in \mathbb{N}$, say for $i \in \{i_0, i_1, \ldots, i_l\}$ we have $U_i = A$. Set $z = \max\{i_0, i_1, \ldots, i_l\}$. Then clearly $\phi((\mathfrak{m}, X)^z) \subset O$. Also for any $(\mathfrak{m}, X)^n$ we have that

$$\phi((\mathfrak{m}, X)^n) = \mathfrak{m}^n \times \mathfrak{m}^{n-1} \cdots \times \mathfrak{m} \times \prod_{i=1}^{\infty} A_i \text{ where } A_i = A$$

which we observe to be open.

Finally we note that A is Hausdorff and complete so $A^{\mathbb{N}}$ is Hausdorff and complete. Then since $A[[X]] \cong A^{\mathbb{N}}$, $A[[X]]$ is Hausdorff and complete. ☐

Lemma 3.26. Suppose $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals of a commutative ring A. Then

(i.) $\operatorname{rad}(\mathfrak{p}^m) = \mathfrak{p}$ for all $m \in \mathbb{N}$.

(ii.) $\operatorname{rad}(\mathfrak{p} + \mathfrak{q}) = \operatorname{rad}(\operatorname{rad}(\mathfrak{p}) + \operatorname{rad}(\mathfrak{q}))$

Lemma 3.27. Let $\mathfrak{p}_i$ for $i \in \{1, \ldots, n\}$ give a set of comaximal primes in A. Then

$$\bigcap_{i=1}^{n} \mathfrak{p}_i = \prod_{i=1}^{n} \mathfrak{p}_i$$

Proof. Follows from the usual intersection equals product proof together with associativity of intersection and product of ideals. ☐

Lemma 3.28 (Chinese remainder theorem)**.** Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$ be comaximal ideals of A (i.e. $\mathfrak{a}_i + \mathfrak{a}_j = A$ for $i \neq j$). Then the map $A \to \prod A/\mathfrak{a}_i$ is surjective with kernel $\mathfrak{a}_1 \cdots \mathfrak{a}_k$

Proof. Construct a map $\phi\colon A \to \prod A/\mathfrak{a}_i$ such that $\phi(a) = (a + \mathfrak{a}_1, \ldots, a + \mathfrak{a}_n)$. It is easy to see that the kernel of this map is $\bigcap \mathfrak{a}_i = \prod \mathfrak{a}_i$ where the equality comes from comaximality and the above lemma. For surjectivity it is enough to show for example there is an element $x \in A$ such that $\phi(x) = (1, 0, \ldots, 0)$. Since $\mathfrak{a}_1 + \mathfrak{a}_i = (1)$ for all $i > 1$ we have equations

$$\begin{aligned} u_1 + v_1 &= 1 \\ u_2 + v_2 &= 1 \\ &\vdots \\ u_n + v_n &= 1 \end{aligned}$$

Let $x = \prod_{i=2}^{n} v_i$ so $x = \prod (1 - u_i) = 1 \equiv 1 \mod \mathfrak{a}_1$ and $x \equiv 0 \mod \mathfrak{a}_i$ for all $i > 1$, as desired. ☐

Proposition 3.29 (Maximal ideal isomorphism of Hausdorff completion)**.** Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ be pairwise distinct maximal ideals of the ring A and let $\tau = \bigcap_{i=1}^{n} \hat{A}_{\mathfrak{m}_i}$. Then there is a canonical isomorphism

$$\hat{A} = \prod_{1 \leqslant i \leqslant k} \hat{A}_{\mathfrak{m}_i}$$

where $\hat{A}$ is the τ-adic completion of A and $\hat{A}_{\mathfrak{m}_i}$ is the Hausdorff completion of $A_{\mathfrak{m}_i}$ for the $\mathfrak{m}_i A_i$-adic topology.

Proof. First note that since $\mathfrak{m}_i$'s are pairwise distinct maximal ideals, they are comaximal. Then we for $i \neq j$ also have that

$$\begin{aligned} \operatorname{rad}(\mathfrak{m}_i^n + \mathfrak{m}_j^n) &= \operatorname{rad}(\operatorname{rad}(\mathfrak{m}_i^n) + \operatorname{rad}(\mathfrak{m}_j^n)) \\ &= \operatorname{rad}(\mathfrak{m}_i + \mathfrak{m}_j) \\ &= \operatorname{rad}((1)) = (1) \end{aligned}$$

Thus $\mathfrak{m}_i^n + \mathfrak{m}_j^n = (1)$ and they are comaximal. Thus their instersection is equal to their product by one of the lemmas above and by the Chinese Remainder Theorem we have that

$$A/\tau^n = A/(\mathfrak{m}_1^n \cap \cdots \cap \mathfrak{m}_k^n) = \prod A/\mathfrak{m}_i^n$$

But each factor in the right hand side is a local ring with only prime and maximal ideal $\mathfrak{m}/\mathfrak{m}_i^n$ by prime correspondence. Then the localization of each factor $A/\mathfrak{m}_i^n$ at $\mathfrak{m}/\mathfrak{m}_i^n$ which is isomorphic to $A_{\mathfrak{m}_i}/\mathfrak{m}_i^n A_{\mathfrak{m}_i}$ is also canonically isomorphic to itself. (note that $\mathfrak{m}_i^n$ localized at $\mathfrak{m}_i$ is just $\mathfrak{m}_i^n A_{\mathfrak{m}_i}$). So we have

$$A/\tau^n = \prod A_{\mathfrak{m}_i}/\mathfrak{m}_i^n A_{\mathfrak{m}_i}$$

We note that $\mathfrak{m}_i^n A_{\mathfrak{m}_i} = (\mathfrak{m}_i A_{\mathfrak{m}_i})^n$ and taking inverse limits of both sides we attain

$$\hat{A} = \prod \hat{A}_{\mathfrak{m}_i}$$

□

Remark 3.30 (Proposition for semilocal rings)**.** Note that the proposition applies to the case of a semilocal ring A where $\mathfrak{m}_i$ are taken to be the maximal ideals of A. Then τ is the radical of A.

4 Graded rings and modules

Definition 4.1 (Graded ring, module)**.** Let A be a ring. A is called a graded ring if it has a direct sum decomposition

$$A = \bigoplus_{n \in \mathbb{Z}} A_n$$

where A_n's are additive subgroups such that $A_n = 0$ for all $n < 0$ and $A_p A_q \subset A_{p+q}$. A module M over a graded ring A is called a graded module if it has a direct sum decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

such that M_n's are additive subgroups such that $M_n = 0$ for all $n < 0$ and $A_p M_q \subset M_{p+q}$.

Definition 4.2 (Graded ring associated to a filtered ring)**.** Let M be a filtered module over the filtered ring A. We define

$$\mathrm{gr}(M) = \bigoplus \mathrm{gr}_n(M) = \bigoplus M_n/M_{n+1}$$

Then given the canonical maps from each $A_q \times M_p$ to M_{p+q} we attain bilinear maps $\mathrm{gr}_q(A) \times \mathrm{gr}_p(M) \to \mathrm{gr}_{p+q}(M)$. Bilinearity follows from the bilinearity of the previous map and we check that these maps are well defined as follows: Let $(a + A_{p+1}, m + M_{q+1}) = (a' + A_{p+1}, m' + M_{q+1})$. Thus $a - a' \in A_{p+1}$ and $m - m' \in M_{q+1}$. So

$$am - a'm' = am - a'm - a'm' + a'm = (a - a')m + a'(m' - m) \in M_{p+q+1}$$

and $am + M_{p+q+1} = a'm' \in M_{p+q+1}$ as desired. These maps induce the bilinear map $\mathrm{gr}(A) \times \mathrm{gr}(M) \to \mathrm{gr}(M)$ in an obvious manner.

This map provides $\mathrm{gr}(M)$ with a $\mathrm{gr}(A)$-module structure and in particular for $M = A$ we attain a map $\mathrm{gr}(A) \times \mathrm{gr}(A) \to \mathrm{gr}(A)$ which makes $\mathrm{gr}(A)$ into a ring.

These structures are called the graded module and ring associated to the filtered ring A and filtered module M, respectively.

Proposition 4.3 (Induced and quotient filtrations on the filtered modules)**.** For a filtered module M and a quotient module M/N with the quotient filtration, the topology induced on M/N is the quotient of that of M.

Proof. We first show that the quotient map is continuous with the induced topology on M/N. Let $V/N \subset M/N$ be open and $x \in f^{-1}(V/N)$. Then $f(x) = v + N \in V/N$ where $v \in V$. But V/N is open so $\exists k \in \mathbb{Z}$ such that $v + N + N_k \subset V/N$ which implies that $v + N + \frac{M_k+N}{N} \subset V/N$. Then since

$$f(x + M_k) \subset v + N + \frac{M_k + N}{N}$$

we have that

$$x + M_k \subset f^{-1}\left(v + N + \frac{M_k + N}{N}\right) \subset f^{-1}(V/N)$$

as desired. □

Example 4.4 (Formal power series in n variables)**.** Let k be a ring and let $A = k[[X_1, \dots, X_r]]$ be the algebra $\mathfrak{m} = (X_1, \dots, X_r)$ and give A the $\mathfrak{m}$-adic filtration. The graded ring $\mathrm{gr}(A)$ associated to A is the polynomial algebra $k[X_1, \dots, X_r]$, graded by total degree.

Set $B = k[X_1, \dots, X_r]$ and let B_n denote the subgroup of finite sums of monomials of total degree n. Note first that $B_n = 0$ for all $n < 0$ and $B_p B_q \subset B_{p+q}$ and $B = \bigoplus_{n\in\mathbb{Z}} B_n$. Also note that $\mathfrak{m}^n$ is generated by monomials of degree n. Moreover, $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ is then the ring of all polynomials of total degree exactly n as if $p \in \mathfrak{m}^n$ has total degree greater than n then $p \in \mathfrak{m}^{n+1}$ and $p \equiv 0 \mod \mathfrak{m}^{n+1}$.

Proposition 4.5. The canonical maps $M \to \hat{M}$ and $A \to \hat{A}$ induce isomorphisms $\mathrm{gr}(M) = \mathrm{gr}(\hat{M})$ and $\mathrm{gr}(A) = \mathrm{gr}(\hat{A})$.

Proof. We have $M/M_{n+1} \cong \hat{M}/\hat{M}_{n+1}$. We first define $\phi\colon M_n/M_{n+1} \to \hat{M}/\hat{M}_{n+1}$ and restrict to the image to get $M_n/M_{n+1} \cong \hat{M}_n/\hat{M}_{n+1}$ where the last isomorphism holds since $\hat{M}_n$ is precisely the completion of M_n. □

Proposition 4.6 (Surjective strict morphism)**.** Let $u\colon M \to N$ be a morphism of filtered modules. We suppose that M is complete, N is Hausdorff, and $\mathrm{gr}(u)$ is surjective. Then u is a surjective strict morphism and N is complete.

Proof. We aim to show that u is a surjective strict morphism and then N is complete as it is isomorphic to a quotient of M. u is already given to be a morphism of filtered modules, i.e., $u(M_n) \subset N_n$ for all $n \in \mathbb{Z}$. Let $n \in \mathbb{Z}$

and $y \in N_n$. We will construct a sequence (x_k) of elements of M_n such that $u(\lim x_n) = y$. To this end we will set

$$x_{k+1} \equiv x_k \mod M_{n+k} \quad (1)$$
$$u(x_k) \equiv y \mod N_{n+k} \quad (2)$$

Let $x_0 = 0$ and suppose x_k has been constructed. Then $u(x_k) - y \in N_{n+k}$ and since $\mathrm{gr}(u)\colon \mathrm{gr}(M) \to \mathrm{gr}(N)$ is surjective, there exists some $t_k \in M_{n+k}$ such that $u(t_k) \equiv u(x_k) - y \mod N_{n+k+1}$. Let $x_{k+1} = x_k - t_k$ so $x_{k+1} - x_k = -t_k \in M_{n+k}$ and $u(x_k - t_k) = u(x_{k+1}) \equiv y \mod N_{n+k+1}$.

Now by (1), we see that (x_k) is Cauchy. Since M is complete, (x_k) converges to at least a point $x \in M$. Since M_n is closed, $x \in M_n$. By definition, for all $k \in \mathbb{Z}$ there exists $s_k \in \mathbb{Z}$ such that $x_n \in x + M_k$ for $n \geqslant s_k$. Then

$$u(x_n) \in u(x) + M_k \text{ for } n \geqslant s_k$$

and thus $u(x_k) \to u(x)$. But by the definition of (x_k), there exists $s'_k \in \mathbb{Z}$ such that

$$u(x_n) \in y + N_k \text{ for } \geqslant s'_k$$

Then $u(x_k) \to y$ but N is Hausdorff so the two limits must be equal and we have $u(x) = y$. Then $u(M_n) = N_n$ and u is a surjective strict morphism. Then N is complete as it is isomorphic to a quotient of complete M. □

Corollary 4.7. Let A be a complete filtered ring, M a Hausdorff filtered A-module, $(x_i)_{i\in I}$ a finite family of elements of M, and (n_i) a finite family of integers such that $x_i \in M_{n_i}$. Let $\overline{x}_i$ be the image of x_i in $\mathrm{gr}_{n_i}(M)$. If the $\overline{x}_i$ generate the $\mathrm{gr}(A)$-module $\mathrm{gr}(M)$, then the x_i generate M, and M is complete.

Proof. Let $E = A^I$ and E_n denote the subgroup of E which consists of elements of the form $(a_i)_{i\in I}$ where $a_i \in A_{n-n_i}$ for each $i \in I$. We check that this really is a filtration:

i. $E_0 = \prod_{i\in I} A_{0-n_i} = A^I$
ii. $E_{n+1} = \prod_{i\in I} A_{n+1=n_i} \subset \prod_{i\in I} A_{n-n_i} = E_n$ since $A_{n+1-n_i} \subset A_{n-n_i}$
iii. $A_q E_q = \prod_{i\in I} A_p A_{q-n_i} \subset \prod_{i\in I} A_{p+q-n_i} = E_{p+q}$

Thus E is a filtered module with filtration (E_n). We note that the filtration induced topology on E is precisely the product topology:

By definition of the product topology, it is the coarsest topology such that the projections are continuous (this is checked in a similar manner to the formal power series isomorphism in the last chapter) and thus must be contained in the filtration topology. Conversely, note that each basic open set U in the filtration topology is of the form $x + E_k$ for some $E_k \in (E_n)$ and $x \in X$. But then U is given by $\prod_{i\in I}(x + A_{k-n_i})$ which is clearly open in the finite product topology. Since it contains the basic open sets, the product topology must contain all their arbitrary unions and thus the entire topology given by (E_n).

Now we define $u\colon E \to M$ where $u((a_i)) = \sum a_i x_i$. This is clearly a morphism of modules. We wish to show that $u(E_n) \subset M_n$.

Let $(a_i)_{i\in I} \in E_n$. So $a_i \in A_{n-n_i}$ for all $i \in I$. Then $u((a_i)) = \sum a_i x_i$ where $a_i \in A_{n-n_i}$ and $x_i \in M_{n_i}$ and thus $a_i x_i \in M_{n-n_i+n_i} = M_n$. So $u((a_i)) \in M_n$ as desired and u is a morphism of filtered modules. Then u induces well defined $\mathrm{gr}(u)\colon \mathrm{gr}(E) \to \mathrm{gr}(M)$ and for any $i \in I$ the sequence (z_j) where $z_j = 0$ if $i \neq j$ and $z_j = 1$ if $i = j$ maps to x_i under u. Since $\overline{x}_i$ generates $\mathrm{gr}(M)$, $\mathrm{gr}(u)$ is surjective. E is complete as a product of complete spaces and M is Hausdorff so by the previous proposition we have the result. □

Remark 4.8. By the result of the previous theorem, u is a surjective strict morphism so $u(E_n) = M_n \implies M_n = \sum A_{n-n_i} x_i$.

Corollary 4.9. If M is a Hausdorff filtered module over the complete filtered ring A, and if $\mathrm{gr}(M)$ is a finitely generated (resp. Noetherian) $\mathrm{gr}(A)$-module, then M is finitely generated (resp. Noetherian, and each of its submodules is closed).

Proof. The first corollary shows that if $\mathrm{gr}(M)$ is finitely generated then M is complete and finitely generated. Now suppose $\mathrm{gr}(M)$ is Noetherian. If N is a submodule of M with the induced filtration then we can associate $\mathrm{gr}(N)$ to a graded $\mathrm{gr}(A)$-submodule of $\mathrm{gr}(M)$.

Let $i\colon N \to M$ be the canonical injection. Then

$$i(N_n) = i(M_n \cap N) = M_n \cap N \subset M_n$$

and so i is a morphism of filtered modules. Then $\mathrm{gr}(i)\colon \mathrm{gr}(N) \to \mathrm{gr}(M)$ is well defined. Moreover, for $\mathrm{gr}(i)_n\colon N_n/N_{n+1} \to M_n/M_{n+1}$ if

$$u(a + (M_{n+1} \cap N)) = a + M_{n+1} = 0 + M_{n+1}$$

then $a \in M_{n+1}$ but $a \in N$ already so $a \in M_{n+1} \cap N$ and $a + (M_{n+1} \cap N) = 0 + (M_{n+1} \cap N)$, showing that $\mathrm{gr}(i)_n$ is an injection. Then the induced map $\mathrm{gr}(i)$ is also an injection and by the first isomorphism theorem $\mathrm{gr}(N) \cong \mathrm{Im}(g(i))$ which is a submodule of $\mathrm{gr}(M)$. Then if $\mathrm{gr}(M)$ is Noetherian, $\mathrm{gr}(N)$ is finitely generated hence N is finitely generated and complete by the previous corollary.

Claim: Complete submodules of Hausdorff modules are closed.

Proof. Suppose N is a submodule of a Hausdorff module and N is not closed. Let $b \in \overline{N} - N$. Construct a sequence (x_n) such that

$$\begin{aligned} x_0 &= z_0 \in (b + M_0) \cap N \\ x_1 &= z_1 \in (b + M_1) \cap N \\ &\vdots \\ x_n &= z_n \in (b + M_n) \cap N \end{aligned}$$

Then clearly (x_n) converges to b and thus it is Cauchy. Note that since (x_n) converges to a point $b \notin N$ and the module is Hausdorff, it cannot converge to a point in N. Thus N cannot be complete, a contradiction. □

Now by the claim we see that N is closed and since N is finitely generated and arbitrary, M is Noetherian. □

Corollary 4.10. Let $\mathfrak{m}$ be an ideal of the ring A. Suppose that $\mathrm{gr}(A)$ is Noetherian, $\mathfrak{m}$ is finitely generated, and A is Hausdorff and complete for the $\mathfrak{m}$-adic topology. Then A is Noetherian.

Proof. Suppose $\mathfrak{m} = (x_1, x_2, \dots, x_r)$. Then

$$\mathrm{gr}_n(A) = 0 \text{ for } n < 0$$
$$\mathrm{gr}_n = \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

Claim: $\mathfrak{m}^n$ is generated by monomials $x_1^{d_1} \cdots x_r^{d_r}$ such that $d_1 + \cdots + d_r = n$.

Proof. Let $p \in \mathfrak{m}$. Then

$$\begin{aligned} p &= \sum_{\text{finite}} (\alpha_{11}x_1 + \cdots + \alpha_{1r}x_r)(\cdots)(\alpha_{n1}x_1 + \cdots + \alpha_{nr}x_r) \\ &= \sum_{\text{finite}} \sum_{d_1+\cdots+d_r=n} k_\bullet x_1^{d_1} \cdots + x_r^{d_r} \end{aligned}$$

and the claim follows. □

Define $\phi\colon (A/\mathfrak{m})[X_1, \dots, X_r] \to \mathrm{gr}(A)$ such that $\overline{\alpha} X_1^{d_1} \cdots X_r^{d_r} \mapsto \overline{\alpha x_1^{d_1} \cdots x_r^{d_r}} \in \mathrm{gr}_n(A)$. We check that this map is well-defined:

Let $\overline{\alpha}_1 X_1^{d_1} \cdots X_r^{d_r} = \overline{\alpha}_2 X_1^{d_1} \cdots X_r^{d_r}$. Thus $\alpha_1 - \alpha_2 \in \mathfrak{m}$. Then

$$(\alpha_1 - \alpha_2) x_1^{d_1} \cdots x_r^{d_r} \in \mathfrak{m}^{n+1}$$

So $\alpha_1 x_1^{d_1} \cdots x_r^{d_r} = \alpha_2 x_1^{d_1} \cdots x_r^{d_r}$.

ϕ is easily verified to be a ring homomorphism and it is surjective since for any $\overline{\alpha x_1^{d_1} \cdots x_r^{d_r}} \in \mathrm{gr}_n(A)$ we have that $\phi(\overline{X}_1^{d_1} \cdots X_r^{d_r}) = \overline{\alpha x_1^{d_1} \cdots x_r^{d_r}}$.

Then $\mathrm{gr}(A)$ is isomorphic to a quotient of $(A/\mathfrak{m})[X_1, \dots, X_r]$ which is Noetherian by Hilbert's basis theorem. Thus $\mathrm{gr}(A)$ is Noetherian as it is the quotient of a Noetherian ring. Then by the previous corollary, A is Noetherian. □

Proposition 4.11. If the filtered ring A is Hausdorff, and if $\mathrm{gr}(A)$ is a domain, then A is a domain.

Proof. Since A is Hausdorff and (A_n) is a neighborhood basis at 0, for nonzero $x, y \in A$ there exists $n, m \in \mathbb{Z}$ such that $x \in A_n - A_{n+1}$ and $y \in A_m - A_{m+1}$ (This can be shown either using the T_1 nature of A or via $\bigcap A_n = 0$).

Then $\overline{x} \neq \overline{0}$ in $A_n/A_{n+1} = \mathrm{gr}(A)$ and $\overline{y} \neq \overline{0}$ in $A_m/A_{m+1} = \mathrm{gr}(A)$. Hence $\overline{x}, \overline{y} \in \mathrm{gr}(A)$ are nonzero.

Since $\mathrm{gr}(A)$ is a domain $\overline{x} \cdot \overline{y} \neq \overline{0} \implies xy \neq 0$. □