All Morphisms Are Equal, But Some Morphisms Are More Equal Than Others

An Introduction to Higher Category Theory

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Preliminaries

Definition (Inaccesible cardinal, Grothendieck universe)

We call a cardinal κ **inaccessible** if the collection of sets $\mathcal{V}_{<\kappa}$ of hereditary cardinality less than κ satisfies the ZFC axioms. $\mathcal{V}_{<\kappa}$ is called a **Grothendieck universe**.

Axiom

We assume the existence of sufficiently many inaccessible cardinals.

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Remark

This axiom is logically independent from ZFC.

2-categories

Definition (Strict 2-categories)

A **strict 2-category** $\mathcal{C}$ consists of:

- Objects, also called 0-morphisms.
- For each pair of objects (A, B), a category $\mathrm{Hom}_{\mathcal{C}}(A, B)$, whose objects are called 1-morphisms and morphisms are called 2-morphisms.
- Composition functors $\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$ that are associative and have identity 1-morphisms.

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Example

$\mathcal{R}\mathrm{el}$	
Objects	Sets
1-morphisms	Relations
2-morphisms	Implications

$\mathcal{C}\mathrm{at}$	
Objects	Categories
1-morphisms	Functors
2-morphisms	Natural transformations

2-categories

Remark

Strict associativity and unitality laws often go against natural constructions in higher category theory. For this reason, we seek to relax the definition of a strict 2-category.

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Definition (Weak 2-category)

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Definition (Weak 2-category)

A **weak 2-category** is a 2-category where associativity and unitality of composition hold only up to natural isomorphism.

Example

Given a topological space X, its fundamental 2-groupoid is the 2-groupoid whose

- objects are the points (elements) of X;
- 1-morphisms are continuous paths $[0,1] \to X$;
- 2-morphisms are homotopies between such paths, fixing their endpoints;
- composition is given by concatenation of paths and homotopies.

2-categories

Theorem

Every strict 2-category is 2-equivalent to a weak 2-category and every weak 2-category is biequivalent to a strict 2-category.

Remark (Recursive definition)

The definition of an n-category could be given in a recursive manner. Although, this unfortunately only works for strict n-categories. To define a weak n-category using the established theory of $(n-1)$*-categories is impossible since we require the associativity laws of* $n-1$ *morphisms to hold only up to n-isomorphism, which is not yet defined.*

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Remark

Higher categories are not as well-behaved. For example, not every weak 3-category is equivalent to a strict 3-category (Consider the fundamental 3-groupoid of S^2*).*

Motivation for ∞-categories

Let X be a topological space and $0 \leqslant n \leqslant \infty$. We can extract a weak n-category $\pi_{\leqslant n}X$.

- 0-morphisms of $\pi_{\leqslant n}X$ are the points of X.
- For $x, y \in \pi_{\leqslant n}X$ a 1-morphism from x to y is a continuous path $[0, 1] \to X$ starting at x and ending at y.
- 2-morphisms are given by homotopies of paths.
- 3-morphisms are given by homotopies of homotopies.

  $\vdots$

In some sort of limit, we hope to arrive at a theory of $(\infty, 0)$-categories, where every morphism is invertible up to homotopy.
We will generalize this notion and come up with a theory of $(\infty, 1)$-categories, where every k-morphism for $k > 1$ is invertible up to homotopy.

Topological construction

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A **topological category** is a category enriched over category $\mathcal{CG}$ of compactly generated and weakly Hausdorff topological spaces. We denote the category of topological categories as $\mathcal{C}\mathrm{at}_{\mathrm{Top}}$.

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Problem

Topological categories have strict associativity and unitality. To stay in the category, we have to **straighten** our morphisms. This process involves converting between various models of homotopy theory, which is highly non-trivial.

Simplicial preliminaries

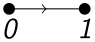
Definition (Category of (combinatorial) simplices)

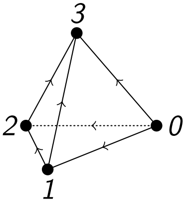
We define a category Δ, called the category of simplices or the **simplex category**, consisting of the following data:

- Objects linearly ordered sets $[n] := \{0, 1, \ldots, n\}$ for every $n \geqslant 0$.
- Morphisms weakly monotone maps, i.e., $f\colon [m] \to [n]$ such that $a \leqslant b$ implies $f(a) \leqslant f(b)$.

Remark

The objects of Δ can be drawn as simplices with ordered vertices. For example, we have

$[0] = \bullet$ $\quad$ $[1] =$ 

$\quad$ $[3] =$ 

Simplicial preliminaries

Definition (Simplicial set)

A functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ with

- Sets $X_n := X([n])$ called n-simplices for every $n \geqslant 0$.
- Maps $d_i^n\colon X_n \to X_{n-1}$ and $s_i^n\colon X_n \to X_{n+1}$ called the i-th face and degeneracy maps, respectively, satisfying the simplicial identities.

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Definition (Set_Δ)

We define the category Set_Δ to be the category of presheaves on Δ, i.e., $\mathrm{Set}_\Delta := \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$. A morphism in this category is just a natural transformation $X \Rightarrow Y$, which amounts to arrows $X_n \to Y_n$ commuting with the face and degeneracy maps.

Simplicial preliminaries

Definition (Standard n-simplex)

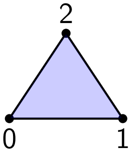
We denote representable functor $\mathrm{Hom}_{\Delta}(-, [n]) = \Delta^n$ and call it the standard n-simplex.

Definition (Horn)

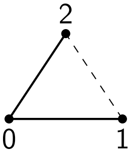
Fix $n \geqslant 0$ and $i \in [n]$. Then the set of all order preserving morphisms $p\colon [m] \to [n]$ such that $p([m]) \cup \{i\} \neq [n]$ is called the i-th horn of Δ^n, and it is denoted by Λ^n_i.

Definition (Boundary)

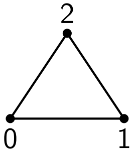
Fix $n \geqslant 0$. Then, the smallest simplicial set containing all faces of Δ^n is called the boundary of Δ^n, and it is denoted by $\partial\Delta^n$.



Standard 2-Simplex



0th Horn



Boundary

Simplicial preliminaries

Definition (Horn fillers, Kan complex)

Let X be a simplicial set and Λ^n_i be the i-th horn of the standard n-simplex with a map $f\colon \Lambda^n_i \to X$. By a **filler** for Λ^n_i we mean a dotted arrow g such that the following diagram commutes:

$$\begin{array}{ccc} \Lambda^n_i & \xrightarrow{f} & X \\ \big\downarrow & \overset{g}{\nearrow} & \\ \Delta^n & & \end{array}$$

As such, we say that X admits a filler for the i-th horn.
A simplicial set is called a **Kan complex** if it admits fillers for all horns.

Simplicial construction

Definition (∞-category)

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Remark

Given an ∞-category $\mathcal{C}$

- *elements of the set $\mathcal{C}_0$ are objects (0-morphisms)*
- *$\mathcal{C}_1$ are morphisms between objects. The two face maps $\mathcal{C}_1 \rightrightarrows \mathcal{C}_0$ define the source and target*
- *For $k > 1$ the elements of $\mathcal{C}_k$ are the higher morphisms.*

Equivalence of constructions

Definition (Geometrical n-simplex)

For $n \geqslant 0$ the geometrical n-simplex $|\Delta^n|$ is given by

$$|\Delta^n| = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; t_i \geqslant 0 \text{ and } \sum_{n=0}^{n} t_i = 1 \right\}$$

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Definition

We define a functor

$$\begin{aligned} |-|\colon \Delta &\to \mathcal{CG} \\ [n] &\mapsto |\Delta^n| \end{aligned}$$

sending each morphism $\phi\colon [n] \to [m]$ to a morphism

$$\begin{aligned} |\phi|\colon |\Delta^n| &\to |\Delta^m| \\ (t_0, \ldots, t_n) &\mapsto (a_0, \ldots, a_m) \text{ where } a_i = \sum_{i \in \phi^{-1}(j)} t_i \end{aligned}$$

Equivalence of constructions

Definition ($\mathrm{Sing}\ X$)

The **geometric realization functor**

$$|-|\colon \mathrm{Set}_\Delta \to \mathcal{CG}$$

is the left Kan extension of $|-|\colon \Delta \to \mathcal{CG}$ along the Yoneda embedding $\Delta \to \mathrm{Set}_\Delta$. It has a right adjoint called the **singular simplicial set functor**:

$$\begin{aligned} \mathrm{Sing}\colon \mathcal{CG} &\to \mathrm{Set}_\Delta \\ X &\mapsto \mathrm{Hom}_{\mathcal{CG}}(|\Delta^{(-)}|, X) \end{aligned}$$

That is, an n-simplex in $\mathrm{Sing}\ X$ is a continuous map $|\Delta^n| \to X$.

Equivalence of constructions

Informal Definition (Model structure)

A **model structure** on category $\mathcal{C}$ is given by sets of weak equivalences, cofibrations, and fibrations required to satisfy some axioms. This construction encodes homotopy data.

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Let

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D} \colon G$$

be an adjunction. It is called a **Quillen adjunction** if G preserves fibrations and F preserves cofibrations.

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Lemma

(Ken Brown's Lemma) Let $F \dashv G$ be a Quillen adjunction. Then G preserves weak equivalences of fibration objects, and F preserves weak equivalences of cofibration objects.

Equivalence of constructions

Definition (Quillen equivalence)

A Quillen adjunction is called a **Quillen equivalence** if its left and right derived functors are equivalences of localized categories.

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Theorem (Quillen)

The adjunction

$$|-|\colon \mathrm{Set}_\Delta \rightleftarrows \mathcal{CG}\colon \mathrm{Sing}$$

is a Quillen equivalence of the Kan-Quillen model structure on Set_Δ *and the classical model structure on* $\mathcal{CG}$*.*

Definition (Functor τ and category τ^n)

We define the functor

$$\begin{aligned} \tau\colon \Delta &\to \mathcal{C}\mathrm{at} \\ [n] &\mapsto \tau^n \end{aligned}$$

where τ^n is the category given by objects $0, 1, 2, \ldots, n$ and

$$\mathrm{Hom}_{\tau^n}(i, j) = \begin{cases} *, & i \leqslant j \\ \varnothing, & i > j \end{cases}$$

One can visualize the category τ^n as a diagram

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

Definition (Nerve)

For a category $\mathcal{C}$, its nerve consists of the following datum.

- Functors $\tau^0 \to \mathcal{C}$ as 0-simplices (i.e. the objects of $\mathcal{C}$)
- Functors $\tau^1 \to \mathcal{C}$ as 1-simplices which correspond to diagrams $c_o \xrightarrow{f} c_1$ in $\mathcal{C}$.
- Functors $\tau^n \to \mathcal{C}$ as n-simplices which correspond to diagrams of length n in $\mathcal{C}$

$$c_o \xrightarrow{f_{01}} c_1 \xrightarrow{f_{12}} c_2 \xrightarrow{f_{23}} \cdots \xrightarrow{f_{(n-1)n}} c_n$$

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This is the right adjoint of

$$\mathrm{Lan}_y \tau \colon \mathrm{Set}_\Delta \to \mathcal{C}\mathrm{at}$$
$$X \mapsto \mathrm{colim}_{\Delta^k \to X} \Delta^k$$

∞-categories out of ordinary categories

Recover $\mathcal{C}$ (up to isomorphism) from its nerve:

- The objects of $\mathcal{C}$ are given by the 0-simplices of the nerve.
- A morphism from an object c_0 to an object c_1 is given by 1-simplex ϕ with $d_0(\phi) = c_1$ and $d_1(\phi) = c_0$.
- For an object c of $\mathcal{C}$, the identity morphism id_c is given by the degenerate simplex $s_0(c)$.
- Finally, given a diagram $c_0 \xrightarrow{\phi} c_1 \xrightarrow{\psi} c_2$ the edge of $\mathrm{N}(\mathcal{C})$ corresponding to $\psi \circ \phi$ is characterized uniquely by the fact that there is a unique $\sigma \in \mathrm{N}(\mathcal{C})_2$ with $d_2(\sigma) = \phi$, $d_0(\sigma) = \psi$, $d_1(\sigma) = \psi \circ \phi$.

The composition and unitality laws can be checked easily from these definitions.

Infinity category of spaces

Proposition

Every $(\infty, 0)$-category is a Kan complex.

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Definition ($\mathcal{K}\mathrm{an}$)

We denote by $\mathcal{K}\mathrm{an}$ the full subcategory of Set_Δ given by the collection of Kan complexes.

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Proposition

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Definition ($\mathcal{S}$)

We call $\mathrm{N}(\mathcal{K}\mathrm{an})$ the ∞-category of spaces and denote it by $\mathcal{S}$.

Proposition

Every ∞-category is enriched over $\mathcal{S}$.

References

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